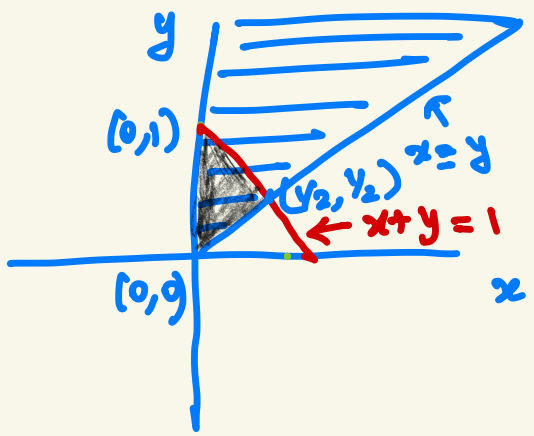
Problems on random vectors & transformations

Ex: Consider the joint pdf

$$f(x,y) = e^{-y} \qquad 0 < x < y < \infty$$
$$= 0 \qquad \text{otherwise}$$



i) Calculate $P(X+Y \geq 1)$

$$P(X+Y \geq 1) = 1 - P(X+Y \leq 1)$$

$$= 1 - \int_0^{y_2} \int_x^{1-x} e^{-y} \, dy \, dx$$

$$= 1 - \int_0^{1/2} \left(e^{-x} - e^{-(1-x)} \right) dx$$

$$= 1 - \left[e^{-1/2} - e^{-(1-1/2)} - (1 - e^{-1}) \right] = 2e^{-1/2} - e^{-1}$$

Marginal of x

Note $f(x,y) = e^{-y}$ $\quad 0 < x < y < \infty$

$= 0$ $\quad$ o.w.

If $x \leq 0$, $f(x,y) = 0$ $\forall y$. $\quad f_x(x) = 0$ for $x \leq 0$.

If $x > 0$, $f(x,y) = e^{-y}$ if $x < y$

$$\Rightarrow f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{x}^{\infty} e^{-y}\,dy = e^{-x}$$

$\Rightarrow$ Marginal of x, $f_x(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$

Conditional of y given $x = x$ (only for $x > 0$)

$$f(y|x) = \frac{f(x,y)}{f_x(x)} = \frac{e^{-y}}{e^{-x}} = e^{-(y-x)} \quad \text{if } y > x$$

$$= 0 \quad \text{if } y < x$$

Conditional mean and variance.

$$E(Y|X=x) = \int_x^{\infty} y\, e^{-(y-x)}\, dy = 1+x$$

$$\text{var}(Y|X=x) = E(Y^2|X=x) - \left[E(Y|X=x)\right]^2$$

$$E(Y^2|X=x) = \int_x^{\infty} y^2\, e^{-(y-x)}\, dy =$$

$$\text{var}(Y|X=x) = 1$$

Ex.2 Transformation of random variables.

$(X,Y) \sim f_{XY}(x,y)$ ← joint pdf of (X,Y)

Let $U = g_1(x,y)$, $V = g_2(x,y)$: Transformation.

Compute joint pdf of (U,V).

$\mathcal{A} = \{(x,y) : f_{XY}(x,y) > 0\}$

$\mathcal{B} = \{(u,v) : u = g_1(x,y),\ v = g_2(x,y) \text{ for } (x,y) \in \mathcal{A}\}$

Then $f_{UV}(u,v)$ is positive on $\mathcal{B}$.

Assume that g_1, g_2 are one-to-one from $\mathcal{A}$ to $\mathcal{B}$ (By def. onto on $\mathcal{B}$)

$\Rightarrow u = g_1(x,y)$ and $v = g_2(x,y)$

$\Rightarrow x = h_1(u,v)$ and $y = h_2(u,v)$

$$J = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

where $\frac{\partial x}{\partial u} = \frac{\partial}{\partial u} h_1(u,v)$, $\frac{\partial y}{\partial u} = \frac{\partial}{\partial u} h_2(u,v)$

$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v} h_1(u,v)$, $\frac{\partial y}{\partial v} = \frac{\partial}{\partial v} h_2(u,v)$

Then $f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v)) |J|$ — on B

$= 0$ — o.w.

Ex: Sum & difference of normal variables.

Let $X, Y \overset{iid}{\sim} n(0,1)$

Consider the transformation

$$U = X + Y, \qquad V = X - Y$$

In the notation above, $g_1(x,y) = x+y$

$g_2(x,y) = x-y$

$$f(x,y) = f_X(x) f_Y(y) \qquad -\infty < x < \infty, \; -\infty < y < \infty$$

$$= \frac{1}{2\pi} e^{-\left(\frac{x^2+y^2}{2}\right)}$$

$\mathcal{A} = \mathbb{R}^2$, to determine $\mathcal{B} \subseteq \mathbb{R}^2$ on which $f_{UV}(u,v)$ is positive.

$u = x+y, \quad v = x-y$

$x = h_1(u,v) = \frac{u+v}{2}$ and $y = h_2(u,v) = \frac{u-v}{2}$

$\Rightarrow B = \mathbb{R}^2$ and g_1, g_2 are one-to-one from A to B.

$$J = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = -\frac{1}{2}$$

$\therefore f_{UV}(u,v) = f_{XY}(h_1(u,v), h_2(u,v)) |J|$ on B

$$= \frac{1}{2\pi} e^{-\frac{1}{2}\left[\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2 \right]} \times \frac{1}{2} \qquad -\infty < u < \infty, \quad -\infty < v < \infty$$

$$f_{UV}(u,v) = \left(\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2}} e^{-u^2/4} \right) \left(\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2}} e^{-v^2/4} \right)$$

$\Rightarrow$ $U = X + Y$ and $V = X - Y$ are independent normal random variables

with $U \sim N(0, 2)$, $V \sim N(0, 2)$

Generalization: Sums and differences of indep. normal random variable are independent as long as $\text{var}(X) = \text{var}(Y)$.

Ex: $U =$

$V =$

Ex: $X, Y \overset{iid}{\sim} n(0,1)$

$U = \frac{X}{Y}$, $V = |Y|$

Note the transformation is **NOT** one-one as (x,y) and $(-x,-y)$ are both mapped to the same point (u,v).

$A_1 = \{(x,y) : y>0\}$; $A_2 = \{(x,y) : y<0\}$

$A_0 = \{(x,y) : y=0\}$

$A = \mathbb{R}^2 = A_0 \cup A_1 \cup A_2$ disjoint partition and

$P(A_0) = 0$ and the transformation is one-to-one on A_1 & A_2.

$B = \{(u,v) : v > 0\}$: Im of both A_1 & A_2.

(∵ $(x,y) \in A_i$ for any i
$v = |y|$ and $u = \frac{x}{y}$ can be any real no. as x is any real no.)

$x = h_1(u,v) = uv$
$y = h_2(u,v) = v$

on A_1 $J = \begin{bmatrix} v & u \\ 0 & 1 \end{bmatrix}$

$x = \tilde{h}_1(u,v) = -uv$
$y = \tilde{h}_2(u,v) = -v$

on A_2 $J = \begin{bmatrix} -v & -u \\ 0 & -1 \end{bmatrix}$

In both these cases $|\det(J)| = v$

$$f_{XY}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \qquad -\infty < x < \infty, \quad -\infty < y < \infty$$

Now

$$f_{UV}(u,v) = \underbrace{\frac{1}{2\pi} e^{-\frac{1}{2}((uv)^2+v^2)} \times |v|}_{\text{for } A_1} + \underbrace{\frac{1}{2\pi} e^{-\frac{1}{2}((-uv)^2+(-v)^2)} \times |v|}_{\text{for } A_2}$$

$$f_{UV}(u,v) = \frac{v}{\pi} e^{-(u^2+1)v^2/2} \qquad -\infty < u < \infty, \quad 0 < v < \infty$$

In order to compute density of $U = X/Y$ we compute marginal density from the joint.

$$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\, dv$$

$$= \int_{0}^{\infty} \frac{v}{\pi} e^{-(u^2+1)v^2/2} \, dv$$

Put $z = v^2 \Rightarrow dz = 2v\,dv \Rightarrow v\,dv = \frac{dz}{2}$

$v = 0 \Rightarrow z = 0$ & $v = \infty \Rightarrow z = \infty$

$$f_U(u) = \frac{1}{2\pi} \int_{0}^{\infty} e^{-(u^2+1)z/2} \, dz$$

$$= \frac{1}{2\pi} \cdot \frac{2}{u^2+1} = \frac{1}{\pi(u^2+1)} \qquad -\infty < u < \infty$$

$\Rightarrow U$ is a Cauchy random variable. ∎

Ex: Insect lays large number of eggs.

Each egg survives with probability 'p.'

How many eggs will survive on average??

large number of eggs is also a random variable.

x = no. of survivors.

y = no. of eggs laid.

$x|y \sim$ binomial (y, p)

$y \sim$ Poisson (λ)

} Hierarchical model.

$$P(X=x) = \sum_{y=0}^{\infty} P(X=x, Y=y)$$

$$= \sum_{y=0}^{\infty} P(X=x \mid Y=y)\, P(Y=y)$$

$$= \sum_{y=x}^{\infty} \binom{y}{x} p^x (1-p)^{y-x} \frac{e^{-\lambda} \lambda^y}{y!}$$

[Conditional prob. is 0 if $y<x$]

$$= \frac{(\lambda p)^x e^{-\lambda}}{x!} \sum_{y=x}^{\infty} \frac{((1-p)\lambda)^{y-x}}{(y-x)!}$$

$$= \frac{(\lambda p)^x e^{-\lambda}}{x!} e^{(1-p)\lambda} = \frac{(\lambda p)^x e^{-\lambda p}}{x!}$$

$\Rightarrow X \sim \text{Poisson}(\lambda p)$

$\Rightarrow E(X) =$ avg. number of survivors

$= \lambda p$

Theorem: X & Y are two random variables.

$$E(X) = E\left(E(X|Y)\right)$$

provided expectations exist.

pf: $$E(X) = \iint x\, f(x,y)\, dx\, dy = \int \left[\int x\, f(x|y)\, dx\right] f_Y(y)\, dy$$

$$= \int \underbrace{E[X|Y=y]}_{h(y)} f_Y(y)\, dy = E\left[E(X|Y)\right]$$

■

We could have used this result in earlier example

$$E(X) = E\left[E(X|Y)\right)$$

$$= E(pY)$$ since $X|Y \sim \text{binomial}(Y, p)$

$$= p\,E(Y)$$

$$= p\lambda$$ since $Y \sim \text{Poisson}(\lambda)$.

Similar result exists for variance.

$$\text{var}(X) = E(\text{var}(X|Y)) + \text{var}(E(X|Y))$$

Multinomial distribution.

R : random expt. with n distinct outcomes with probabilities $p_1, p_2, \ldots, p_n$.

m indep. trials of R are performed.

Let x_i = number of times i^{th} outcome occurred.

$(x_1, x_2, \ldots, x_n)$: discrete random vector

$$p(x_1 = x_1, x_2 = x_2, \ldots, x_n = x_n)$$

$$= f(x_1, \ldots, x_n) = \frac{m!}{x_1!\, x_2! \cdots x_n!}\, p_1^{x_1} \cdots p_n^{x_n}$$

where $(x_1, \ldots, x_n)$ is a vector of nonnegative integers such that $\sum_{i=1}^{n} x_i = m$.

[Multinomial distribution with m trials are cell probabilities equal to $p_1, p_2, \ldots, p_n$].

Why pmf?? (Multinomial - thm).

Let $m, n \in \mathbb{Z}_+$. $\mathcal{A}$ be the set of vectors $(x_1, \cdots, x_n)$ s.t. $x_i \in \mathbb{Z}_+ \cup \{0\}$ and $\sum_{i=1}^{n} x_i = m$. Then for any real nos. $a_1, a_2, \cdots, a_n$

$$(a_1 + a_2 + \cdots + a_n)^m = \sum_{x \in \mathcal{A}} \frac{m!}{x_1! \cdots x_n!} a_1^{x_1} \cdots a_n^{x_n}$$

Marginal of x_n.

Denote by $\mathcal{B}$ the set of all $(x_1, \cdots, x_{n-1})$ where x_i's are non-negative integers and $\sum_{i=1}^{n-1} x_i = m - x_n$

for a fixed $x_n \in \{0, 1, \cdots, m\}$

$$f(x_n) = \sum_{(x_1,\dots,x_{n-1})\in B} \frac{m!}{x_1!\cdots x_n!}\, p_1^{x_1}\cdots p_n^{x_n}$$

$$= \sum_{(x_1,\dots,x_{n-1})\in B} \frac{m!}{x_1!\cdots x_n!}\, p_1^{x_1}\cdots p_n^{x_n}\, \frac{(m-x_n)!(1-p_n)^{m-x_n}}{(m-x_n)!(1-p_n)^{m-x_n}}$$

$$= \frac{m!}{x_n!(m-x_n)!}\, p_n^{x_n}(1-p_n)^{m-x_n} \times \sum_{(x_1,\dots,x_{n-1})\in B} \frac{(m-x_n)!}{x_1!\cdots(x_{n-1})!} \left(\frac{p_1}{1-p_n}\right)^{x_1}\cdots\left(\frac{p_{n-1}}{1-p_n}\right)^{x_{n-1}}$$

Note that $p_1+\cdots+p_n=1$ & $x_1+\cdots+x_{n-1}=m-x_n$ ⇒ the sum is 1

$$\Rightarrow f(x_n) = \frac{m!}{x_n!(m-x_n)!}\, p_n^{x_n}(1-p_n)^{m-x_n} \qquad x_n = 0,1,2,\dots,m$$

$X_n \sim$ Binomial (m, p_n)

Conditional given x_n

$$f(x_1, \cdots, x_{n-1} \mid x_n) = \frac{f(x_1, \cdots, x_n)}{f(x_n)}$$

$$= \frac{\frac{m!}{x_1! \cdots x_n!} p_1^{x_1} \cdots p_n^{x_n}}{\frac{m!}{x_n!(m-x_n)!} p_n^{x_n} (1-p_n)^{m-x_n}}$$

$$= \frac{(m-x_n)!}{x_1! \cdots x_{n-1}!} \left(\frac{p_1}{1-p_n}\right)^{x_1} \cdots \left(\frac{p_{n-1}}{1-p_n}\right)^{x_{n-1}}$$

This is multinomial distribution with $(m-x_n)$ trials with cell probabilities $\frac{p_1}{1-p_n}, \cdots, \frac{p_{n-1}}{1-p_n}$ resp.

Ex: Consider a hierarchical model:

$Y|X \sim$ Poisson (X) $\longrightarrow f_{Y|X=x} = \dfrac{e^{-x}\, x^{y}}{y!}$, $y = 0, 1, 2, \ldots$

& $X \sim$ gamma (α, λ)

$$f_X(x) = \begin{cases} \dfrac{\lambda^{\alpha}}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\lambda x} & ; \quad x > 0 \\ 0 & ; \quad \text{o.w.} \end{cases}$$

$$f_Y(y) = \int_0^{\infty} f_{Y|X}(y|x)\, f_X(x)\, dx$$

$$= \int_0^{\infty} \frac{e^{-x}\, x^{y}}{y!}\, \frac{\lambda^{\alpha}}{\Gamma(\alpha)}\, x^{\alpha-1} e^{-\lambda x}\, dx$$

$$= \frac{\lambda^\alpha}{y!\,\Gamma(\alpha)} \int_0^\infty x^{y+\alpha-1} e^{-(\lambda+1)x} \, dx$$

$$= \frac{\lambda^\alpha}{y!\,\Gamma(\alpha)} \frac{\Gamma(y+\alpha)}{(\lambda+1)^{y+\alpha}}$$

If α is positive integer,

$$= \frac{\lambda^\alpha}{y!\,(\alpha-1)!} \frac{(y+\alpha-1)!}{(\lambda+1)^{y+\alpha}}$$

$$f_Y(y) = \binom{y+\alpha-1}{y} \left(\frac{\lambda}{\lambda+1}\right)^\alpha \left(\frac{1}{\lambda+1}\right)^y$$

$Y \sim$

Ex: $X_i \mid p_i \sim \text{Bernoulli}(p_i) \quad i = 1, 2, \cdots, n$

$p_i \sim \text{beta}(\alpha, \beta)$

(Arise in a situation where measuring success of a drug on n patients. Since each patient gives different response to the drug, each patient will have different probability of success).

$p_i \sim \text{Beta}(\alpha, \beta)$

$$\Rightarrow f_{p_i}(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} \qquad 0 \le p \le 1$$

$$= 0 \qquad \text{o.w.}$$

Define $Y = \sum_{i=1}^{n} X_i$ = total no. of successes.

i) Show $E(y) = n\frac{\alpha}{\alpha+\beta}$

$$E(y) = E\left(\sum x_i\right)$$

$$= \sum_{i=1}^{n} E(x_i)$$

$$E(x_i) = E\left(E(x_i \mid p_i)\right) = E(p_i) = \frac{\alpha}{\alpha+\beta}$$

$$\therefore \quad E(y) = n\frac{\alpha}{\alpha+\beta}$$

$$\text{var}(y) = \text{var}\left(\sum x_i\right) = \sum_{i=1}^{n} \text{var}(x_i)$$

(Note: $\text{var}(x_i) = \text{var}\left(E(x_i \mid p_i)\right) + E\left(\text{var}(x_i) p_i\right)$

$$= \text{var}(p_i) + E(p_i(1-p_i))$$

$$=$$