Let $x_1, x_2, \cdots, x_n$ be a random sample from $f(\theta)$.

i.e. $x_1, \cdots, x_n \overset{iid}{\sim} f(\theta)$

Notation: $X_1, \cdots, X_n$: random variables

$x_1, \cdots, x_n$: realizations of the r.v.s $X_1, \cdots, X_n$ resp.

Ex: Let $x_1, \cdots, x_{10}$ be a random sample from Bernoulli(p)

Then $x_1, \cdots, x_{10} \overset{iid}{\sim}$ Bernoulli(p)

$x_1, \cdots, x_{10} \equiv 1, 0, 1, 1, 1, 0, 0, 1, 0, 0$ ← } different realizations of $x_1, \cdots, x_n$

$x_1, \cdots, x_{10} \equiv 1, 1, 0, 0, 0, 0, 0, 1, 1, 0$ ← }

General objective:

Data is observed / random sample is observed.

$x_1, \dots, x_n$

We know $x_1, \dots, x_n$ is a random sample from $f(\theta)$ with realizations $x_1, \dots, x_n$.

Objective: From $x_1, \dots, x_n$, guess (estimate) θ.

In the case of $x_1, \dots, x_{10} \overset{iid}{\sim}$ Bernoulli (p)

1, 1, 0, 0, 0, 0, 0, 1, 1, 0

Guess of p, denoted as $\hat{p}$, will be $\frac{4}{10}$.

" $\frac{\sum x_i}{10}$: this is the realization of the random variable $\frac{\sum_{i=1}^{10} x_i}{10}$.

Thus different functions of random sample may be of importance in estimating the parameters of functions of parameters.

Let $T(X_1, \ldots, X_n)$ be a function of random sample. It is called as statistic and the distribution of $T(X_1, \ldots, X_n)$ is called as sampling distribution of T.

Consider the case of $X_1, \ldots, X_n$ a random sample from $N(\mu, \sigma^2)$. Here we assume that μ is unknown and σ^2 is known. From the random sample, $X_1, \ldots, X_n$, we want to estimate the mean μ.

Clearly, $T(x_1, \cdots, x_n) = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$ will help.

Note that $E(\bar{x}) = \mu$

$$\text{var}(\bar{x}) = \frac{\sigma^2}{n}$$

In fact, we have proved $\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$

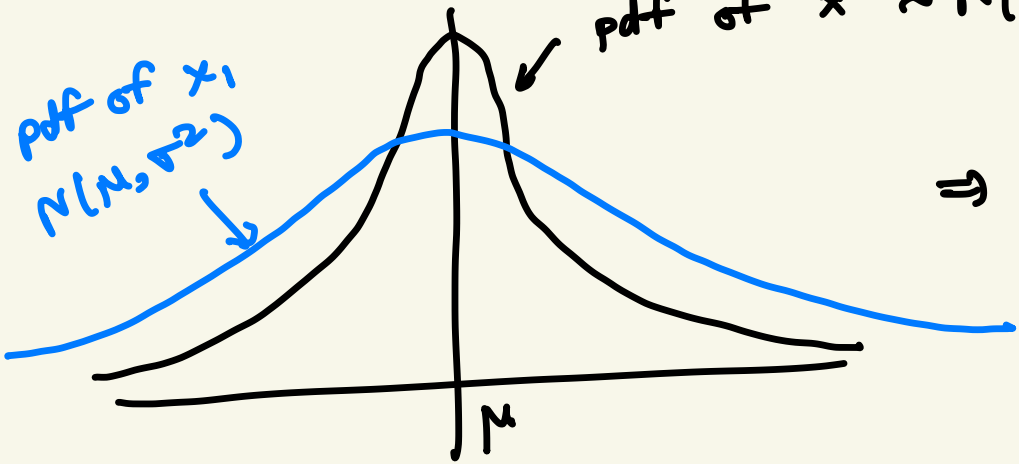
In this case, the statistic $T(x_1, \cdots, x_n) = \bar{x}$ is used to estimate μ and hence called as an estimator of μ. The realization of the the estimator is called as an estimate is a real value in this case.

Ex: Let $-2.9, 3.1, 1.7, -2.2, 0.9, -0.08, 0.7$ be a random sample from $N(\mu, 1)$. Find the estimate of μ??

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_7}{7}$$; estimater

$$\bar{x} = \frac{-2.9 + 3.1 + 1.7 - 2.2 + 0.9 - 0.08 + 0.7}{7}$$

↑ estimate of μ from the data.



$$\Rightarrow \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

... standardization.

pdf of $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

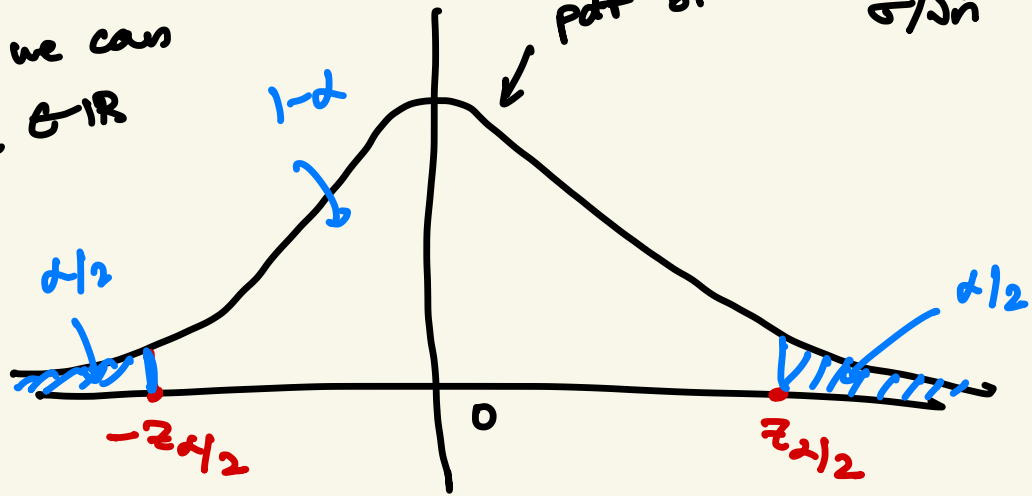
Given $\alpha \in [0,1]$, we can find point $z_{\alpha/2} \in \mathbb{R}$ s.t.

$$P(|Z| \geq z_{\alpha/2}) = \alpha$$

or

$$P(|Z| \leq z_{\alpha/2}) = 1-\alpha$$

$$P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) = 1-\alpha$$



Now consider the case when $X_1, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$. (Both μ and σ^2 are unknown).

We want to estimate σ^2

Estimator of $\mu = \bar{X} = \frac{\sum_{i=1}^{n} x_i}{n}$.

Estimator of $\sigma^2 = s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$.

Result: Let $x_1, \ldots, x_n$ be any real nos. given and

$\bar{x} = \frac{\sum x_i}{n}$ and $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

Prove: $\min_{a \in \mathbb{R}} \sum_{i=1}^{n} (x_i - a)^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2$

P:

$$\sum_{i=1}^{n}(x_i-a)^2 = \sum_{i=1}^{n}(x_i-\bar{x}+\bar{x}-a)^2$$

$$= \sum_{i=1}^{n}(x_i-\bar{x})^2 + 2(\bar{x}-a)\sum_{i=1}^{n}(x_i-\bar{x})$$

$$+ \sum_{i=1}^{n}(\bar{x}-a)^2$$

$$\Rightarrow \sum_{i=1}^{n}(x_i-a)^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2 + \sum_{i=1}^{n}(\bar{x}-a)^2$$

■

Result: s^2 and $\bar{x}$ are independent.

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$= \frac{1}{n-1} \left[(x_1 - \bar{x})^2 + \sum_{i=2}^{n} (x_i - \bar{x})^2 \right]$$

$$= \frac{1}{n-1} \left[\left(\sum_{i=2}^{n} (x_i - \bar{x}) \right)^2 + \sum_{i=2}^{n} (x_i - \bar{x})^2 \right]$$

$\left[\because \sum_{i=1}^{n} x_i - \bar{x} = 0 \right]$

Thus s^2 is function of $x_2 - \bar{x}, \ldots, x_n - \bar{x}$.

We want to show that $x_2 - \bar{x}, \ldots, x_n - \bar{x}$ are independent of $\bar{x}$.

The joint pdf of $x_1, \dots, x_n$ is

$$f(x_1, \dots, x_n) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{\sum x_i^2}{2}}$$

Why assume $x_1, \dots, x_n$ iid $N(0,1)$

Consider the transformation:

$$y_1 = \bar{x}, \quad y_2 = x_2 - \bar{x}, \quad \dots, \quad y_n = x_n - \bar{x}$$

Then

$$\det \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & & & \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & 1-\frac{1}{n} \end{bmatrix}$$

$$= \frac{1}{n}$$

$\bar{x} = y_1, \; x_2 = y_2 + y_1, \dots, x_n = y_n + y_1$

$\Rightarrow \frac{x_1 + \cdots + x_n}{n} = y_1 \Rightarrow x_1 = n y_1 - x_2 - \cdots - x_n \Rightarrow x_1 = y_1 - y_2 - \cdots - y_n$

$$\therefore f(y_1,\dots,y_n) = \frac{n}{(2\pi)^{n/2}} e^{-\frac{1}{2}(y_1 - y_2 \cdots - y_n)^2} \times$$

$$e^{-\frac{1}{2}\left(\sum_{i=2}^{n}(y_i + y_1)^2\right)}$$

Note that

$$(y_1 - y_2 \cdots - y_n)^2 + \sum_{i=2}^{n}(y_i + y_1)^2$$

$$= n y_1^2 + \sum_{i=2}^{n} y_i^2 + \left(\sum_{i=2}^{n} y_i\right)^2$$

$$\Rightarrow f(y_1,\dots,y_n) = \left[\left(\frac{n}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}n y_1^2}\right] \times \left[\frac{\sqrt{n}}{(2\pi)^{n-1/2}} \times e^{-\frac{1}{2}\left(\sum_{i=2}^{n} y_i^2 + \left(\sum_{i=2}^{n} y_i\right)^2\right)}\right]$$

$\Rightarrow f(y_1, \ldots, y_n) = f(y_1)\, f(y_2, \ldots, y_n)$

$\Rightarrow Y_1$ is indep. of $Y_2, \ldots, Y_n$

$\Rightarrow \bar{x}$ is independent of $x_2 - \bar{x}, \ldots, x_n - \bar{x}$

$\Rightarrow \bar{x}$ is independent of s^2.

Result: $E(s^2) = \sigma^2$

Pf: $E\left(\sum_{i=1}^{n} (x_i - \bar{x})^2\right) = E\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right)$

$= \sum_{i=1}^{n} E(x_i^2) - n E(\bar{x}^2)$

$E\left(\sum_{i=1}^{n} (x_i - \bar{x})^2\right) = n(\sigma^2 + \mu^2) - n\left(\mu^2 + \frac{\sigma^2}{n}\right)$

$= (n-1)\sigma^2$

$$E(S^2) = E\left(\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2\right) = \frac{1}{n-1}E\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)$$

$$= \frac{1}{n-1}(n-1)\sigma^2$$

$$\boxed{E(S^2) = \sigma^2}$$: unbiased estimator of σ^2.

$$Y = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$$: Biased estimator of σ^2.

$$E(Y) = \frac{n-1}{n}\sigma^2$$

1) $X \sim \text{gamma}(\alpha, \lambda)$

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \qquad x > 0$$

$$= 0 \qquad \text{o.w.}$$

2) Recall $X \sim N(0,1)$, $Y = X^2$ then $Y \sim \text{gamma}(\frac{1}{2}, \frac{1}{2})$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-y/2} \qquad y > 0$$

$$= 0 \qquad \text{o.w.}$$

3) Recall if $X_i \sim \text{gamma}(\alpha_i, \lambda)$ for $i = 1, 2, \sim, n$ are indep.

$Y = \sum X_i \sim \text{gamma}\left(\sum_{i=1}^{n} \alpha_i, \lambda\right)$.

Let $x_1, x_2, \ldots, x_n$ be a random sample from $N(0,1)$

$y = x_1^2 + x_2^2 + \ldots + x_n^2 \sim \text{gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$

χ^2_n density with n degrees of freedom.

i.e. χ^2_n is obtained by adding squares of n iid $n(0,1)$ r.v.s. $\quad y \sim \chi^2_n$

$$f_y(y) = \frac{1}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}\, y^{\frac{n}{2}-1}\, e^{-y/2} \qquad y > 0$$

$$= 0 \qquad \text{o.w.}$$

MGF of Y

$$M_Y(t) = \frac{1}{(1-2t)^{n/2}} \qquad t < \frac{1}{2}$$

Result: $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

Here $x_1, \ldots, x_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Proof:

$$\sum_{i=1}^{n} (x_i - \mu)^2 = \sum_{i=1}^{n} (x_i - \bar{x} + \bar{x} - \mu)^2$$

$$= \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2$$

Dividing by σ^2 on both sides.

$$\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2 = \sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{\sigma}\right)^2 + \left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2$$

$$\frac{x_i-\mu}{\sigma} \sim N(0,1)$$

$$\Rightarrow \left(\frac{x_i-\mu}{\sigma}\right)^2 \sim \chi^2_1 \qquad \Rightarrow \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2 \sim \chi^2_n$$

Because $x_1, \ldots, x_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Further, $\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$$\Rightarrow \left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2_1$$

Let $U = \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^2 \sim \chi^2_n$

$V = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\sigma^2} = \frac{(n-1)s^2}{\sigma^2}$

$W = \left(\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2_1$

Note that s^2 & $\bar{x}$ are indep. $\Rightarrow$ V & W are indep.

$U = V + W$

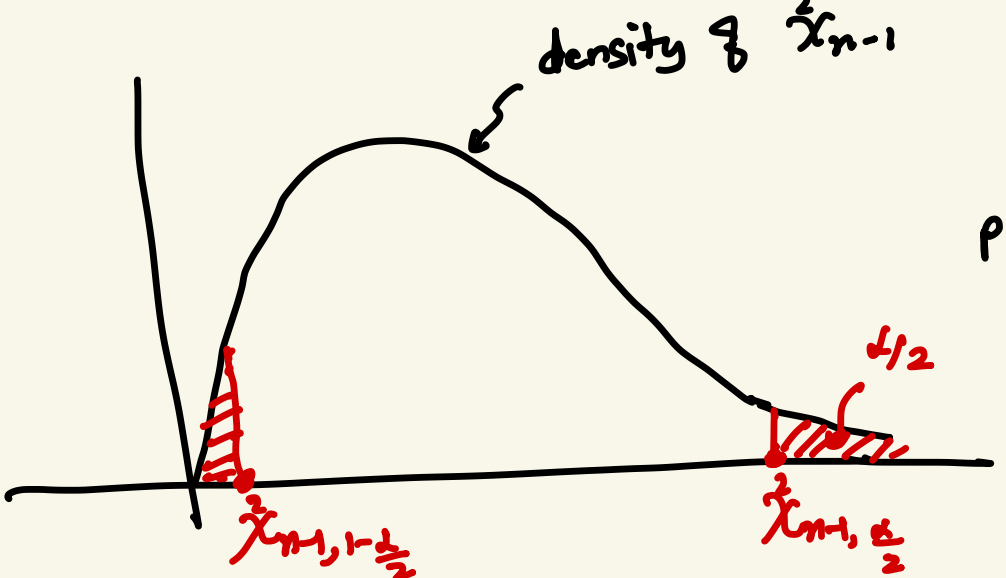
$\Rightarrow M_U(t) = M_{V+W}(t) = M_V(t)\, M_W(t)$

$\Rightarrow \frac{1}{(1-2t)^{n/2}} = M_V(t) \frac{1}{(1-2t)^{1/2}} \Rightarrow M_V(t) = \frac{1}{(1-2t)^{\frac{n-1}{2}}}$

$\Rightarrow V \sim \text{gamma}\left(\frac{n-1}{2}, \frac{1}{2}\right)$

$\Rightarrow V \sim \chi^2_{n-1}$

$\Rightarrow \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$ ∎



If $V \sim \chi^2_{n-1}$, then

$$P\left(\chi^2_{n-1,1-\frac{\alpha}{2}} \le V \le \chi^2_{n-1,\frac{\alpha}{2}}\right) = 1-\alpha$$

i.e.

$$P\left(\chi^2_{n-1,1-\frac{\alpha}{2}} \le \frac{(n-1)s^2}{\sigma^2} \le \chi^2_{n-1,\frac{\alpha}{2}}\right) = 1-\alpha$$

for some $\alpha \in (0,1)$; choose percent points $\chi^2_{n-1,\alpha/2}$

Now consider the case $x_1, x_2, \ldots, x_n$ is a random sample from $N(\mu, \sigma^2)$ and both μ & σ^2 are unknown. We want to estimate μ from the random sample.

We can use the estimator $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$ to estimate μ.

However, note $\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$

$$\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

However here σ is unknown.

* $\bar{x}$: can be measured / calculated from the sample.
  σ^2 : known
  Then could have estimated μ from $\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$.

However, since σ^2 is also unknown, let us use unbiased estimator of σ^2.

Consider the following random variable

$$\frac{\bar{X} - \mu}{S/\sqrt{n}}$$

Q: What is the distribution of this r.v.?

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{\bar{X} - \mu / \sigma/\sqrt{n}}{S/\sigma} = \frac{\bar{X} - \mu / \sigma/\sqrt{n}}{\sqrt{S^2/\sigma^2}}$$

Note $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$, $\frac{S^2}{\sigma^2} \sim \frac{\chi^2_{n-1}}{n-1}$ and the numerator and the denominator are independent.

Therefore, given that $U \sim N(0,1)$, $\frac{V}{p} \sim \chi^2_p$ and U & V are independent, we want to compute the distribution of the random variable $T = \frac{U}{\sqrt{V/p}}$. Here T is called as student's t random variable with p degrees of freedom & $T \sim t_p$ with pdf

$$f_T(t) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \frac{1}{(p\pi)^{1/2}} \frac{1}{\left(1 + \frac{t^2}{p^2}\right)^{\frac{p+1}{2}}} \quad ; -\infty < t < \infty$$

(symmetric density).

Let $U \sim N(0,1)$ & $\frac{V}{p} \sim \chi^2_p$ and U, V are independent. Then the joint density of U & V is given by:

$$f_{U,V}(u,v) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \quad \frac{1}{2^{p/2}\Gamma(\frac{p}{2})} v^{\frac{p}{2}-1} e^{-v/2}$$

$$-\infty < u < \infty, \quad 0 < v < \infty$$

Consider the transformation,

$$t = \frac{u}{\sqrt{v/p}}, \quad w = v$$

Then $v = w$, $u = t\sqrt{\frac{w}{p}}$

$$\frac{\partial v}{\partial w} = 1, \quad \frac{\partial v}{\partial t} = 0 ; \quad \frac{\partial u}{\partial t} = \sqrt{\frac{w}{p}}, \quad \frac{\partial u}{\partial w} = \frac{t}{\sqrt{p}} \cdot \frac{1}{2\sqrt{w}}$$

$$\det J = \det \begin{bmatrix} \sqrt{\frac{w}{p}} & \frac{1}{t\sqrt{p}} \frac{1}{2\sqrt{w}} \\ 0 & 1 \end{bmatrix} = \left(\frac{w}{p}\right)^{1/2}$$

$$\therefore \quad f_{T,W}(t,w) = f_{U,V}\left(t\sqrt{\frac{w}{p}}, w\right)\left(\frac{w}{p}\right)^{1/2}$$

$$= \frac{1}{(2\pi)^{1/2}} e^{-t^2 w/2p} \frac{1}{2^{p/2}\Gamma\left(\frac{p}{2}\right)} w^{p/2-1} e^{-w/2} \left(\frac{w}{p}\right)^{1/2}$$

$$f_T(t) = \int_0^\infty f_{T,W}(t,w)\, dw$$

$$= \frac{1}{(2\pi)^{1/2}} \frac{1}{2^{p/2}\Gamma\left(\frac{p}{2}\right)} \int_0^\infty e^{-t^2 w/2p} w^{p/2-1} e^{-w/2} \left(\frac{w}{p}\right)^{1/2} dw$$

$$f_T(t) = \frac{1}{(2\pi)^{1/2}\, 2^{p/2}\, \Gamma(p/2)\, p^{1/2}} \int_0^\infty e^{-\frac{1}{2}(1+t^2/p)\omega}\, \omega^{p/2+1/2-1}\, d\omega$$

λ $\qquad$ α

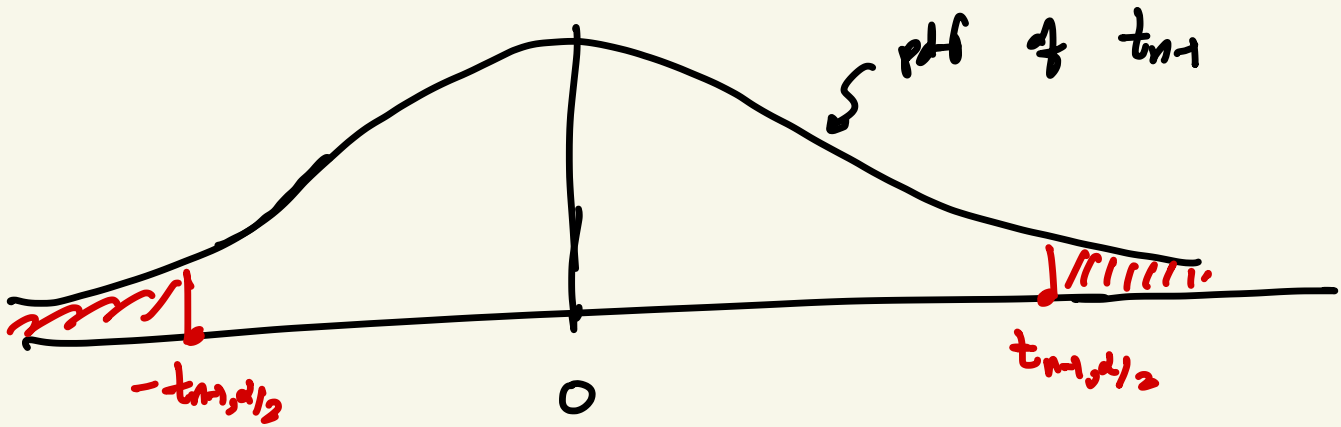
" $\frac{\Gamma(\alpha)}{\lambda^\alpha}$

$$= \frac{1}{(2\pi)^{1/2}\, 2^{p/2}\, \Gamma(p/2)\, p^{1/2}} \cdot \frac{\Gamma\left(\frac{p+1}{2}\right)}{\left(\frac{1}{2}\right)^{\frac{p+1}{2}} \left(1+\frac{t^2}{p}\right)^{\frac{p+1}{2}}}$$

$$f_T(t) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \frac{1}{\sqrt{\pi p}} \frac{1}{\left(1+\frac{t^2}{p}\right)^{\frac{p+1}{2}}} \quad ; -\infty < t < \infty$$

$E[T_p] = 0$, if $p > 1$, $\quad \text{var}(T_p) = \frac{p}{p-2}$ if $p > 2$

T_p has moments upto order $p-1$.



$$\frac{\bar{x}-\mu}{s/\sqrt{n}} \sim t_{n-1}$$

$$\Rightarrow \quad P\left(-t_{n-1,\alpha/2} \leq \frac{\bar{x}-\mu}{s/\sqrt{n}} \leq t_{n-1,\alpha/2}\right) = 1-\alpha$$

Consider the random sample

0.3, −1.8, 1.9, −0.3, 0.01, 0.1 from $N(\mu, \sigma^2)$

Test the hypothesis $H_0: \mu = 1$

$H_1: \mu \neq 1$

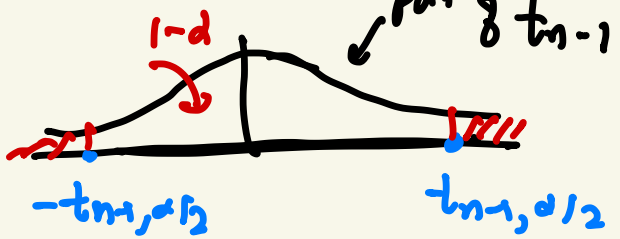
at 0.1 level of significance.

Under H_0 (If H_0 were true)

$$\frac{\bar{x} - 1}{s/\sqrt{n}} \sim t_{n-1}$$

Here $n = 6$

$\bar{x}$, s; to be computed from the data.



$H_0: \sigma^2 = 1$

$H_1: \sigma^2 \neq 1$

$\alpha = 0.1$

$$(n-1)\frac{S^2}{\sigma^2} \sim \chi^2_{n-1}$$

$(n-1)S^2 \sim \chi^2_{n-1}$, then

If use H_0 is true:

$$\chi^2_{n-1,1-\alpha/2} \leq (n-1)S^2 \leq \chi^2_{n-1,\alpha/2}$$

$\Rightarrow$ Fail to reject H_0

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if $(n-1)S^2 \in \left[\chi^2_{n-1,1-\alpha/2}, \chi^2_{n-1,\alpha/2}\right]^C$

$(n-1)S^2 = 7.032$

$\Rightarrow$ fail to reject H_0

Testing of hypothesis

Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$

Case (i) μ unknown, σ^2 known

$H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0$

Test statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

Under H_0, (i.e. if H_0 were true)

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

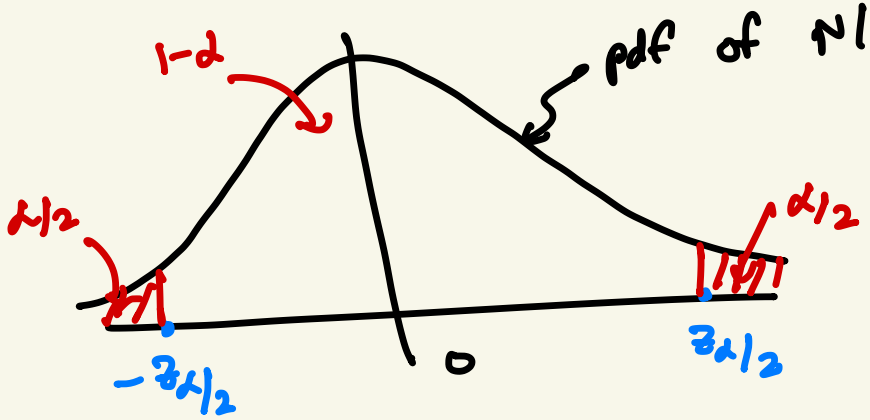
	H_0 is accepted	H_0 is rejected
H_0 is true	✓	✗ Type I
H_1 is true	✗ Type II	✓

$P(H_0 \text{ rejected} \mid H_1 \text{ true}) = \alpha$
= level of significance

$P(H_0 \text{ is failed to reject} \mid H \text{ is true}) = \beta$

$1 - \beta$ = Power of the test

At the level of significance α



If $-z_{\alpha/2} \le \dfrac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \le z_{\alpha/2}$: fail to reject H_0 (accept H_0)

$\left|\dfrac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right| > z_{\alpha/2}$: reject H_0

Evaluation of the test statistic $Z = \dfrac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$ from the given data.

ii) Testing variance.

$X_1, \ldots, X_n$ random sample from $N(\mu, \sigma^2)$

$H_0: \sigma^2 = \sigma_0^2$

$H_1: \sigma^2 \neq \sigma_0^2$

Test statistic : $\frac{(n-1)S^2}{\sigma_0^2}$ where $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

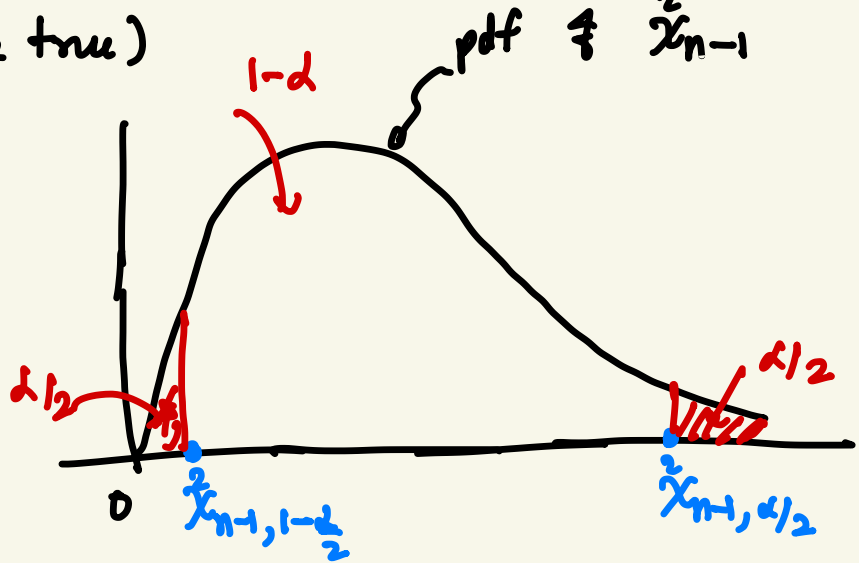
Under H_0 (i.e. if H_0 were true)

$\frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2_{n-1}$

If $\frac{(n-1)S^2}{\sigma_0^2} \in (\chi^2_{n-1,1-\frac{\alpha}{2}}, \chi^2_{n-1,\alpha/2})$

: fail to reject H_0

else Reject H_0



iii) Testing of mean when the variance is unknown.

$X_1, X_2, \cdots, X_n$ is a random sample from $N(\mu, \sigma^2)$.

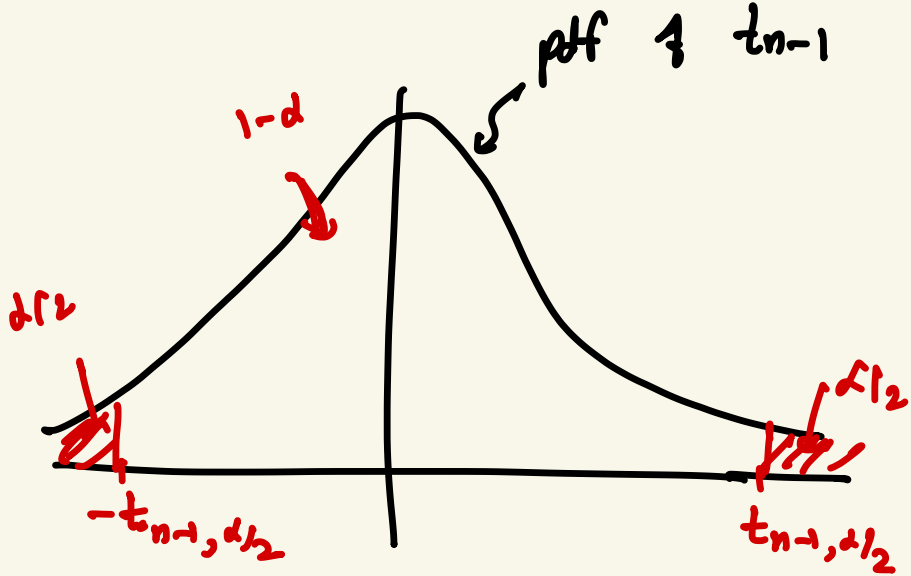
(σ^2 : unknown)

$H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0$

Test statistic: $\dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$

Under H_0: $\dfrac{\bar{x} - \mu_0}{s/\sqrt{n}} \sim t_{n-1}$



At the level of significance α,

if $\dfrac{\bar{x} - \mu_0}{s/\sqrt{n}} \in (-t_{n-1,\alpha/2},\ t_{n-1,\alpha/2})$

: fail to reject H_0

else $\dfrac{\bar{x} - \mu_0}{s/\sqrt{n}} \in (-t_{n-1,\alpha/2},\ t_{n-1,\alpha/2})^c \Rightarrow$ reject H_0.

Definition: The F- distribution with p_1, p_2 degrees of freedom is the distribution of the random variable

$$Y = \frac{U_1/p_1}{U_2/p_2} \qquad \text{where} \quad U_1 \sim \chi^2_{p_1} \quad \text{and} \quad U_2 \sim \chi^2_{p_2}$$

and U_1 & U_2 are independent.

We denote $Y \sim F_{p_1, p_2}$

How to compute pdf of Y?

$$y = \frac{u_1/p_1}{u_2/p_2} \qquad \text{and} \qquad w = u_2 \quad (*)$$

write the joint density of U_1, U_2. Using the transformation (*) write down the joint density of Y, W. Find the marginal density of Y.

Let $X_1, \cdots, X_{n_1}$ be a random sample from $N(\mu_1, \sigma_1^2)$ obtained independently from a random sample $Y_1, Y_2, \cdots, Y_{n_2}$ from $N(\mu_2, \sigma_2^2)$.

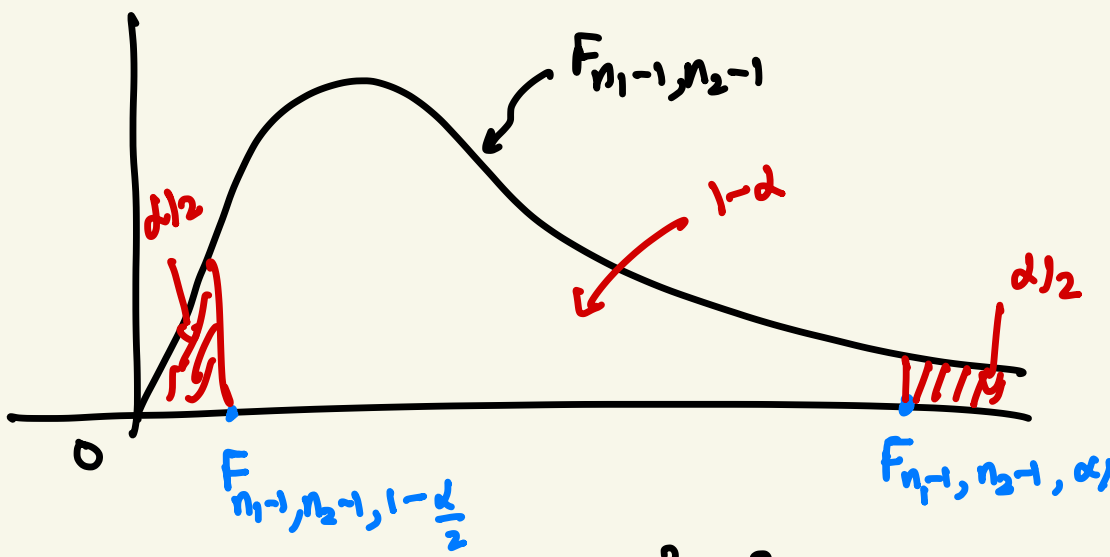
$$\bar{x} = \frac{\sum_{i=1}^{n_1} x_i}{n_1} \quad \text{and} \quad \bar{Y} = \frac{\sum_{i=1}^{n_2} Y_i}{n_2}$$

$$S_1^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (x_i - \bar{x})^2 \quad \text{and} \quad S_2^2 = \frac{1}{n_2 - 1} \sum_{i=1}^{n_2} (Y_i - \bar{Y})^2$$

Note: $\frac{(n_1 - 1) S_1^2}{\sigma_1^2} \sim \chi^2_{n_1 - 1}$ and $\frac{(n_2 - 1) S_2^2}{\sigma_2^2} \sim \chi^2_{n_2 - 1}$

and these r.v.s are independent.

$$\Rightarrow \left[\frac{(n_1 - 1) S_1^2}{\sigma_1^2} \Big/ (n_1 - 1) \right] \Big/ \left[\frac{(n_2 - 1) S_2^2}{\sigma_2^2} \Big/ (n_2 - 1) \right] = \frac{S_1^2 / \sigma_1^2}{S_2^2 / \sigma_2^2} \sim F_{n_1 - 1, n_2 - 1}$$



$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n_1-1,n_2-1}$$

$$\Rightarrow P\left(F_{n_1-1,n_2-1,1-\alpha/2} \leq \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \leq F_{n_1-1,n_2-1,\alpha/2} \right) = 1-\alpha$$

$$\Rightarrow P\left(F_{n_1-1,n_2-1,1-\alpha/2} \frac{S_2^2}{S_1^2} \leq \frac{\sigma_2^2}{\sigma_1^2} \leq F_{n_1-1,n_2-1,\alpha/2} \frac{S_2^2}{S_1^2} \right) = 1-\alpha$$

$$\Rightarrow P\left(\frac{S_1^2}{S_2^2} \cdot \frac{1}{F_{n_1-1,n_2-1,\alpha/2}} \leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{S_1^2}{S_2^2} \cdot \frac{1}{F_{n_1-1,n_2-1,1-\alpha/2}} \right) = 1-\alpha$$

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